

Sustainability in the Stochastic Ramsey Model

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Abstract

In this paper we provide a self-contained exposition of the problem of sustaining a constant consumption level in a Ramsey model. Our focus is on the case in which the output capital-ratio is random. After a brief review of the known results on the probabilities of sustaining a target consumption from an initial stock, we present some new results on estimating the probabilities by using Chebyshev inequalities. Some numerical calculations for these estimates are also provided.

1 Introduction

The discrete time one-good model with a linear production function (“Ramsey Model” in Dorfman-Samuelson-Solow [3, Chapter 11.2] or McFadden [6, Section 6]) has long been a convenient framework for exploring many themes in intertemporal economics. In this paper, the model is used to throw light on issues related to *sustainable consumption*. First, let us pose the sustainability problem in the deterministic case. The economy starts with a positive initial stock x of a good (any reproducible resource or asset: the metaphorical “corn” of growth theory). From this a positive quantity c is subtracted. The parameter c is a datum: it is a target consumption level that the economy wishes to *sustain*. If the remainder $i = x - c$ is zero or negative, the

economy is “ruined.” If the remainder is strictly positive, it is interpreted as an *input* into some productive activity (i.e., an “investment”). The *output* of this activity (or, the *returns* from the investment) is then the stock at the beginning of the next period and is given by $X_1 = r \cdot i = r \cdot (x - c)$, where $r > 0$ is *also* a parameter (“output-capital ratio” in the literature on planning, or an index of “productivity” of investment). Again, in period one, the parameter c is subtracted from X_1 , and the story is repeated. Let N be the first period, if any, such that $X_N < 0$. If N is finite, we say that the economy can *sustain* c up to (but not including) the period N (or, that the economy *survives* up to period N). If N is infinite (i.e., $X_n \geq 0$ for all n), we say that the consumption target c is *sustainable* (or, the economy *survives* forever). There are other interpretations of the model. For example, at a microeconomic level, an economic agent or unit (an investor, a gambler, a firm engaged in managing a fishery,...) is ruined (goes bankrupt, loses the privilege of participating in a game of chance, faces a problem of extinction of the resource managed,...) if its wealth (or, cash reserve, or the stock of the renewable resource,...) X_n in any period falls below some prescribed level c (a minimal rate of dividend, the fee to participate in the game of chance, the target level of harvesting,...). The objective is to study conditions on the parameters r , x and c that determine sustainability. It is not difficult to see that if $r \leq 1$, then no initial x can sustain any $c > 0$. If $r > 1$, the economy can sustain $c > 0$ if and only if $x \geq [r/(r - 1)]c$.

To extend the scope of our analysis, suppose that the returns from the investment are uncertain rather than deterministic. We model this by introducing an i.i.d. sequence ϵ_n of positive random variables. An investment i_n generates output X_{n+1} according to the rule $X_{n+1} = (\epsilon_{n+1}) \cdot i_n$. As in the deterministic case, the economy starts with an initial stock x , and has a target consumption c . It is ruined if $(x - c) \leq 0$. If $x - c > 0$, then the stock in period one is $X_1 = \epsilon_1(x - c)$. Again, if $X_1 - c \leq 0$, it is ruined. Otherwise, after consumption, $X_1 - c$ is invested to generate $X_2 = \epsilon_2 \cdot (X_1 - c)$. In general, one studies the process;

$$X_0 = x, X_{n+1} = (\epsilon_{n+1})(X_n - c)_+, \text{ where } a_+ = \max(a, 0). \quad (1)$$

If $x > c$, the probability of sustaining c is defined as

$$\rho(x) = P(X_n > c \text{ for all } n \geq 0 | X_0 = x). \quad (2)$$

It is shown that

$$\rho(x) = P \left\{ \sum_{n=1}^{\infty} (\epsilon_1 \epsilon_2 \dots \epsilon_n)^{-1} < (x/c) - 1. \right\} \quad (3)$$

This formula (3) can be used to identify conditions on the common distribution of ϵ_n under which the value of $\rho(x)$ can be specified (see Propositions 3.1 and 3.3). For example, if $E \log \epsilon_1 \leq 0$, then $\rho(x) = 0$ for all x and c . The case $E \log \epsilon_1 > 0$ is perhaps the most interesting and turns out to be challenging. Note if we define the random variable Z as:

$$Z = \sum_{n=1}^{\infty} (\epsilon_1 \epsilon_2 \dots \epsilon_n)^{-1}, \quad (4)$$

we realize that the distribution of Z is crucial in determining $\rho(x)$. To this effect, we derive a recursive relation that facilitates computing the moments of Z (Proposition 3.3 and its corollaries). Next, one obtains estimates of survival and ruin probabilities by using Chebyshev's inequalities (section 3.3). We also address the question of estimating the probabilities of sustaining a consumption target up to a finite N (see section 3.5). Numerical calculations for these estimates are provided in sections 3.4 and 3.5.

Our exposition draws upon Majumdar and Radner [5] and Bhattacharya and Waymire [1]. There is a substantial literature using continuous time models that deals with closely related issues. Majumdar and Radner [4] derived the survival probability of an agent in a diffusion model and also extended the analysis to the case in which the agent can sequentially choose from a set of available technologies. Radner [7] provided a review of subsequent research on survival of firms.

Turning to models of mathematical biology, the problem of ruin (extinction) has been investigated in a variety of contexts (see Brauer and Castillo-Chavez [2, Chapters 1, 2]). A particularly celebrated example of “constant yield harvesting” from a population the growth of which is governed by the logistic law leads to the differential equation;

$$dx/dt = \theta x (1 - x/K) - c, \quad (5)$$

where $\theta > 0$ is the “intrinsic” growth rate, and K is the “carrying capacity of the environment” (or, the maximum population size that can be sustained by

the environment. A complete treatment of the extinction and sustainability is available (Brauer and Castillo-Chavez [2, pp.28-29]).

2 Remarks on the Deterministic Case

We make a few remarks on the deterministic case and state the main results. Here, starting with an initial $x > 0$, the economy is ruined if $x - c \leq 0$. If $x > c$, the investment $i_0 = x - c$ generates the stock $X_1 = r \cdot i_0 = r(x - c)$ at the beginning of period 1. The economy is ruined in period 1 if $X_1 - c \leq 0$. If $X_1 > c$, the investment $i_1 = X_1 - c$ generates $X_2 = r \cdot i_1 = r(X_1 - c)$ and so on. If the economy can sustain c up to (but not including) period 2, we know that

$$\begin{aligned} c + i_0 &= x, \\ c + i_1 &= r \cdot i_0. \end{aligned} \tag{6}$$

It follows that $c(1 + 1/r) + i_1/r = x$, leading to:

$$c(1 + 1/r) < x.$$

Hence, if the economy can sustain c up to period N , we must have

$$\sum_{n=0}^{N-1} (1/r^n) < x/c. \tag{7}$$

We immediately conclude that if $r \leq 1$, then for any $c > 0$, there is no $x > 0$ such that c can be sustained forever. Indeed, it is also easy to verify the following:

Proposition 2.1. *Let $r > 1$. The economy can sustain $c > 0$ if and only if*

$$r/(r - 1) \leq x/c. \tag{8}$$

3 The Stochastic Ramsey Model

3.1 Infinite-Horizon Survival Probability and Conditions on the Common Distribution of ϵ_n

To avoid undue repetition, the economy starts with an initial stock $x > 0$ and plans to sustain a consumption level $c > 0$. If $x - c \leq 0$, it is ruined

“immediately.” So focus on the case $x > c$. The investment $i_0 = x - c$ gives rise to the stock in period 1, $X_1 = \epsilon_1(x - c)$, where ϵ_1 is a nonnegative random variable. If $X_1 \leq c$, the economy is ruined in period 1; otherwise, if $X_1 > c$, the investment generates the stock in the next period, and so on. In general, we study:

$$X_0 = x, \quad X_{n+1} = \epsilon_{n+1}(X_n - c)_+ \quad (n \geq 0), \quad a_+ = \max(a, 0), \quad (9)$$

where $\{\epsilon_n : n \geq 1\}$ is an i.i.d. sequence of nonnegative random variables. The state space may be taken to be $[0, \infty)$ with *absorption* at 0. The *probability of survival* of the economic agent with an initial stock $x > c$ is

$$\rho(x) := P(X_n > c \text{ for all } n \geq 0 | X_0 = x). \quad (10)$$

Suppose $P(\epsilon_1 > 0) = 1$. For otherwise, it is simple to check that that eventual ruin is certain, i.e. $\rho(x) = 0$. From (9), successive iteration yields

$$\begin{aligned} X_{n+1} > c &\quad \text{iff } X_n > c + \frac{c}{\epsilon_{n+1}} \quad \text{iff } X_{n-1} > c + \frac{c+c/\epsilon_{n+1}}{\epsilon_n} \dots \\ &\quad \text{iff } X_0 \equiv x > c + \frac{c}{\epsilon_1} + \frac{c}{\epsilon_1\epsilon_2} + \dots + \frac{c}{\epsilon_1\epsilon_2\dots\epsilon_{n+1}}. \end{aligned}$$

Hence, on the set $\{\epsilon_n > 0 \text{ for all } n\}$,

$$\begin{aligned} \{X_n > c \text{ for all } n\} &= \left\{x > c + c \sum_{j=1}^n \frac{1}{\epsilon_1\epsilon_2\dots\epsilon_j} \text{ for all } n\right\} \\ &= \left\{x > c + c \sum_{n=1}^{\infty} \frac{1}{\epsilon_1\epsilon_2\dots\epsilon_n}\right\} = \left\{\sum_{n=1}^{\infty} \frac{1}{\epsilon_1\epsilon_2\dots\epsilon_n} < \frac{x}{c} - 1\right\} \end{aligned}$$

In other words,

$$\rho(x) = P\left\{\sum_{n=1}^{\infty} \frac{1}{\epsilon_1\epsilon_2\dots\epsilon_n} < \frac{x}{c} - 1\right\}. \quad (11)$$

For completeness, we review conditions on the common distribution of ϵ_n under which one has (a) $\rho(x) = 0$, (b) $\rho(x) = 1$, or (c) $\rho(x) < 1$ ($x > c$). The Strong Law of Large Numbers gives

$$\frac{1}{n} \sum_{r=1}^n \log \epsilon_r \xrightarrow{a.s.} E \log \epsilon_1.$$

Thus, if $E \log \epsilon_1 < 0$, $\epsilon_1 \epsilon_2 \cdots \epsilon_n \rightarrow 0$ a.s. This implies that the infinite series in (11) diverges a.s., i.e.,

$$\rho(x) = 0 \quad \text{for all } x \text{ if } E \log \epsilon_1 < 0. \quad (12)$$

If ϵ_1 is non-degenerate and $E \log \epsilon_1 = 0$, then $\sum_{r=1}^n \log \epsilon_r$ also has a subsequence converging to $-\infty$ a.s., and again the series in (11) diverges and $\rho(x) = 0$, $\forall x > c$. By Jensen's Inequality, $E \log \epsilon_1 \leq \log E \epsilon_1$, with strict inequality if ϵ_1 is nondegenerate, which we assume. Therefore, $E \epsilon_1 \leq 1$ implies $E \log \epsilon_1 < 0$, so that $\rho(x) = 0$. Next, let us consider the case $E \log \epsilon_1 > 0$. Define $m := \inf\{z \geq 0 : P(\epsilon_1 \leq z) > 0\}$. We will show that

$$\rho(x) < 1 \quad \text{for all } x, \text{ if } m \leq 1. \quad (13)$$

Fix $A > 0$, however large. One can find n_0 such that $n_0 > A \prod_{r=1}^{\infty} (1 + r^{-2})$ as $\prod (1 + r^{-2}) < \exp\{\sum r^{-2}\} < \infty$. If $m \leq 1$ then $P(\epsilon_1 \leq 1 + r^{-2}) > 0$ for all $r \geq 1$. Consequently,

$$\begin{aligned} 0 &< P(\epsilon_r \leq 1 + r^{-2} \text{ for } 1 \leq r \leq n_0) \\ &\leq P\left(\sum_{r=1}^{n_0} \frac{1}{\epsilon_1 \epsilon_2 \cdots \epsilon_r} \geq \sum_{r=1}^{n_0} \frac{1}{\prod_{j=1}^r (1 + 1/j^2)}\right) \\ &\leq P\left(\sum_{r=1}^{n_0} \frac{1}{\epsilon_1 \epsilon_2 \cdots \epsilon_r} \geq \frac{n_0}{\prod_{j=1}^{\infty} (1 + 1/j^2)}\right) \\ &\leq P\left(\sum_{r=1}^{n_0} (\epsilon_1 \epsilon_2 \cdots \epsilon_r)^{-1} > A\right) \leq P\left(\sum_{r=1}^{\infty} (\epsilon_1 \epsilon_2 \cdots \epsilon_r)^{-1} > A\right). \end{aligned}$$

Since A is arbitrary, (11) is less than 1 for all x , proving (13).

One may also prove that, for $m > 1$,

$$\rho(x) \begin{cases} < 1 & \text{if } x < c(\frac{m}{m-1}), \\ = 1 & \text{if } x \geq c(\frac{m}{m-1}). \end{cases} \quad (m > 1). \quad (14)$$

Observe that $\sum_{n=1}^{\infty} (\epsilon_1 \epsilon_2 \cdots \epsilon_n)^{-1} \leq \sum_{n=1}^{\infty} m^{-n} = 1/(m-1)$, with probability 1 if $m > 1$. The second relation in (14) is subsequently drawn by (11). For the first relation in (14) to be shown, letting $x < cm/(m-1) - c\delta$ for some

$\delta > 0$ implies $x/c - 1 < 1/(m - 1) - \delta$. One can choose $n(\delta)$ such that $\sum_{r=n(\delta)}^{\infty} m^{-r} < \delta/2$ and then choose $\delta_r > 0$ ($1 \leq r \leq n(\delta) - 1$) such that

$$\sum_{r=1}^{n(\delta)-1} \frac{1}{(m + \delta_1) \cdots (m + \delta_r)} > \sum_{r=1}^{n(\delta)-1} \frac{1}{m^r} - \frac{\delta}{2}.$$

Then

$$\begin{aligned} 0 &< P(\epsilon_r < m + \delta_r \text{ for } 1 \leq r \leq n(\delta) - 1) \leq P\left(\sum_{r=1}^{n(\delta)-1} \frac{1}{\epsilon_1 \cdots \epsilon_r} > \sum_{r=1}^{n(\delta)-1} \frac{1}{m^r} - \frac{\delta}{2}\right) \\ &\leq P\left(\sum_{r=1}^{\infty} \frac{1}{\epsilon_1 \cdots \epsilon_r} > \sum_{r=1}^{\infty} \frac{1}{m^r} - \delta\right) = P\left(\sum_{r=1}^{\infty} \frac{1}{\epsilon_1 \cdots \epsilon_r} > \frac{1}{m-1} - \delta\right). \end{aligned}$$

For $\delta > 0$ small enough, the last probability is smaller than $P(\sum(\epsilon_1 \cdots \epsilon_r)^{-1} > x/c - 1)$ if $x/c - 1 < 1/(m - 1)$, i.e., if $x < cm/(m - 1)$. For such x , $1 - \rho(x) > 0$. The desired result is obtained. The following proposition summarizes the above results:

Proposition 3.1. (*[1], [5]*) Let $m := \inf\{z \geq 0 : P(\epsilon_1 \leq z) > 0\}$.

(a) If $E \log \epsilon_1 \leq 0$, then $\rho(x) = 0$ for all x and c .

(b) If $E \log \epsilon_1 > 0$, then

$$\rho(x) = \begin{cases} < 1 & \text{if } m \leq 1 \ \forall x, \text{ or } x < c \frac{m}{m-1} \ (m > 1), \\ = 1 & \text{if } x \geq c \frac{m}{m-1} \ (m > 1). \end{cases}$$

The subsequent Proposition 3.2 provides more explicit statements on $\rho(x)$. This allows $\rho(x)$ to be constructed by estimating Z with the common distribution of ϵ_n .

Proposition 3.2. (*[1], [5]*) Assume $E \log \epsilon_1 > 0$, $Z := \sum_{1 \leq n < \infty} (\epsilon_1 \epsilon_2 \cdots \epsilon_n)^{-1}$. Define $d_1 = \inf\{z \geq 0 : P(Z \leq z) > 0\}$, $d_2 = \sup\{z \geq 0 : P(Z \geq z) > 0\}$, and $M = \sup\{z \geq 0 : P(\epsilon_1 \geq z) > 0\}$. Then,

(a) Z is finite (almost surely).

(b)

$$\rho(x) = \begin{cases} 0 & \text{if } x < c(d_1 + 1), \\ \in (0, 1) & \text{if } c(d_1 + 1) < x < c(d_2 + 1), \\ 1 & \text{if } x > c(d_2 + 1). \end{cases}$$

(c) $\rho(x) = 0$ if $x < \frac{cM}{M-1}$ ($1 < M < \infty$), or for all x ($M \leq 1$).

(d) One can express the (essential) lower bound d_1 and upper bound d_2 of Z in terms of those of ϵ_1 , namely, m and M :

(i) $d_1 = \sum_{1 \leq n < \infty} M^{-n} = 1/(M-1)$ if $M > 1$, and $d_1 = \infty$ if $M \leq 1$.

(ii) $d_2 = \sum_{1 \leq n < \infty} m^{-n} = 1/(m-1)$ if $m > 1$, and $d_2 = \infty$ if $m \leq 1$, where m is defined in (13).

Proof. (a) By the Strong Law of Large Numbers, $(\sum_{1 \leq j \leq n} \log \epsilon_j)/n \rightarrow \mu$ (with probability 1), where $\mu = E \log \epsilon_1 (> 0)$. Therefore, there exists a random variable N which is finite a.s. such that $(\sum_{1 \leq j \leq n} \log \epsilon_j)/n > \mu/2$ for all $n > N$. In other words, $(\epsilon_1 \epsilon_2 \cdots \epsilon_n)^{-1} < e^{-n\mu/2}$ for $n > N$. This suggests that

$$Z = \sum_{1 \leq n \leq N} (\epsilon_1 \epsilon_2 \cdots \epsilon_n)^{-1} + \sum_{n > N} (\epsilon_1 \epsilon_2 \cdots \epsilon_n)^{-1} \quad (15)$$

$$< \sum_{1 \leq n \leq N} (\epsilon_1 \epsilon_2 \cdots \epsilon_n)^{-1} + \sum_{n > N} e^{-n\mu/2} < \infty \text{ a.s.} \quad (16)$$

(b) $x < c(d_1 + 1)$ implies $x/c - 1 < d_1$. One can find $\theta > 0$ such that $x/c - 1 < d_1 - \theta$, which implies

$$\rho(x) = P(Z \leq x/c - 1) \leq P(Z < d_1 - \theta) = 0.$$

Likewise, $x > c(d_2 + 1)$ indicates $x/c - 1 > d_2$. Again, one can find $\theta > 0$ such that $x/c - 1 > d_2 + \theta$, which indicates

$$1 = P(Z < d_2 + \theta) \leq P(Z \leq x/c - 1) = \rho(x).$$

Finally, $c(d_1 + 1) < x < c(d_2 + 1)$ suggests $d_1 < x/c - 1 < d_2$. Then $x/c - 1 > d_1 + \theta$ for some $\theta > 0$, which suggests $\rho(x) = P(Z \leq x/c - 1) \geq P(Z < d_1 + \theta) > 0$. Similarly, $x/c - 1 < d_2 - \theta'$ for some $\theta' > 0$, which turns out to be $\rho(x) = P(Z \leq x/c - 1) \leq P(Z < d_2 - \theta') < 1$.

(c) $x < cM/(M-1)$ can be rewritten in the form of $x/c - 1 < 1/(M-1)$. Notice that $P(\epsilon_1 > M) = 0$. This is because if $P(\epsilon_1 > M) > 0$, then $P(\epsilon_1 \leq M) < 1$. Thus, there exists $\theta > 0$ such that $P(\epsilon_1 \geq M + \theta) > 0$, contradiction. $Z = \sum_{1 \leq n < \infty} (\epsilon_1 \epsilon_2 \cdots \epsilon_n)^{-1} \geq \sum_{1 \leq n < \infty} M^{-n} = 1/(M-1) > x/c - 1$, so that $\rho(x) = P(Z \leq x/c - 1) = 0$.

Next, $M \leq 1$ leads to $\epsilon_1 \epsilon_2 \cdots \epsilon_n \leq 1$ for all n . Then, $(\epsilon_1 \epsilon_2 \cdots \epsilon_n)^{-1} \geq 1$ for all n , which implies $Z = \infty$ almost surely, and $\rho(x) = P(Z \leq x/c - 1) = 0$, no matter how large x may be.

(d)-(i) For $M \leq 1$, $P(\epsilon_n \leq M) = 1$ for all n , yielding $Z = \sum_{1 \leq n < \infty} (\epsilon_1 \epsilon_2 \cdots \epsilon_n)^{-1} \geq \sum_{1 \leq n < \infty} M^{-n} = \infty$ almost surely. It follows that $d_1 = \infty$.

For $M > 1$, again, $P(\epsilon_n \leq M) = 1$ for all n , suggesting $Z = \sum_{1 \leq n < \infty} (\epsilon_1 \epsilon_2 \cdots \epsilon_n)^{-1} \geq \sum_{1 \leq n < \infty} M^{-n} = 1/(M-1)$, almost surely. Therefore, $d_1 \geq 1/(M-1)$. To prove $d_1 \leq 1/(M-1)$, note that there exists $\theta > 0$ such that $M - \theta > 1$, and $P(\epsilon_1 > M - \theta) > 0$ by the definition of M . Since ϵ_n 's are independent, $P(\epsilon_n > M - \theta \text{ for all } n = 1, 2, \dots, N) = \prod_{1 \leq n \leq N} P(\epsilon_n > M - \theta) > 0$ for every N . This implies $P(\sum_{1 \leq n \leq N} (\epsilon_1 \epsilon_2 \cdots \epsilon_n)^{-1} < \sum_{1 \leq n \leq N} (M - \theta)^{-n}) > 0$ for every N . Besides, $\sum_{1 \leq n \leq N} (\epsilon_1 \epsilon_2 \cdots \epsilon_n)^{-1} \rightarrow Z$, and $\sum_{1 \leq n \leq N} (M - \theta)^{-n}$ converges to $1/(M - \theta - 1)$ as $N \rightarrow \infty$. It turns out that $P(Z \leq 1/(M - \theta - 1)) > 0$. Therefore, $d_1 \leq 1/(M - \theta - 1)$ for every $\theta > 0$. Letting $\theta \downarrow 0$ gives rise to $d_1 \leq 1/(M - 1)$.

(d)-(ii) For $m > 1$, $P(\epsilon_1 \geq m) = 1$, indicating $Z = \sum_{1 \leq n < \infty} (\epsilon_1 \epsilon_2 \cdots \epsilon_n)^{-1} \leq \sum_{1 \leq n < \infty} m^{-n} = 1/(m-1)$ almost surely. It follows $d_2 \leq 1/(m-1)$. Note that $P(Z \geq 1/(m-1) + \theta') = 0$ for all $\theta' > 0$. To prove $d_2 \geq 1/(m-1)$, one obtains $P(\epsilon_1 < m + \theta) > 0$ for any $\theta > 0$ and by the definition of m . Arguing as in (i), one demonstrates that $P(\sum_{1 \leq n \leq N} (\epsilon_1 \epsilon_2 \cdots \epsilon_n)^{-1} > \sum_{1 \leq n \leq N} (m + \theta)^{-n}) > 0$ for every N , and $P(Z \geq 1/(m + \theta - 1)) > 0$ as $N \rightarrow \infty$. This proves that $d_2 \geq 1/(m + \theta - 1)$. This is true for every $\theta > 0$, so that $d_2 \geq 1/(m - 1)$.

Now let $m \leq 1$. For every $\theta > 0$, $P(\epsilon_1 \leq 1 + \theta) > 0$, implying $P(Z \geq \sum_{1 \leq n \leq N} (1 + \theta)^{-n}) > 0$. Since $Z = \sum_{1 \leq n < \infty} (\epsilon_1 \epsilon_2 \cdots \epsilon_n)^{-1} \geq \sum_{1 \leq n \leq N} (1 + \theta)^{-n} \rightarrow 1/\theta$ as $N \rightarrow \infty$. Hence, $P(Z \geq 1/\theta) > 0$ for every $\theta > 0$, implying $d_2 = \infty$. \square

3.2 Recursive Computation of the Moments of Z

The following novel method using a recursive relation facilitates computing the moments of Z whose distribution is quite intractable.

Proposition 3.3. *One has the relation*

$$Z = (1/\epsilon_1)(1 + W), \quad (17)$$

where $W = \sum_{2 \leq n \leq N} (\epsilon_2 \epsilon_3 \cdots \epsilon_n)^{-1}$ has the same distribution as Z , and W and ϵ_1 are independent.

Corollary 3.4. *Let $E(\log \epsilon_1) > 0$. (i) If $m = 0$, then $d_2 = \infty$, and (ii) if $M = \infty$, then $d_1 = 0$. In both cases, $0 < \rho(x) < 1$ for $\forall x > c$.*

Proof. (i) $Z > 1/\epsilon_1$, which exceeds any large value with positive probability. (ii) $(1/\epsilon_1)(1 + W)$ approaches zero as ϵ_1 goes to infinity. Now we use Proposition 3.2. \square

Corollary 3.5. *Let $E(\log \epsilon_1) > 0$. Denote the moments of Z and $1/\epsilon_1$ by $\beta_r = EZ^r$, $\gamma_r = E(1/\epsilon_1)^r$, respectively ($r = 1, 2, \dots$). Then, for all r such that $\gamma_r < 1$,*

$$\begin{aligned} \beta_r &= \gamma_r \sum_{0 \leq j \leq r} \binom{r}{j} \beta_j; \quad (1 - \gamma_r) \beta_r = \gamma_r \sum_{0 \leq j \leq r-1} \binom{r}{j} \beta_j; \\ \beta_r &= [\gamma_r / (1 - \gamma_r)] \sum_{0 \leq j \leq r-1} \binom{r}{j} \beta_j. \end{aligned} \quad (18)$$

If $\gamma_r \geq 1$ for some r , $\beta_r = \infty$.

Proof. This relation is derived directly from the representation in Proposition 3.3 by independence of W and ϵ_1 and by the binomial formula for $(1 + W)^r$. \square

Example 3.1. (Lognormal) Let $\epsilon_1 = e^N$ be lognormal, where N is a Normal random variable with mean $\mu > 0$, and a positive variance σ^2 . Applying Corollary 3.4 gives $d_1 = 0$ and $d_2 = \infty$. Hence, by Proposition 3.2, $0 < \rho(x) < 1$ for every $x > c$. $1/\epsilon_1 = e^{-N}$ is also lognormal, $-N$ being Normal

with mean $-\mu < 0$ and variance σ^2 . Hence, the moments of $1/\epsilon_1$ are given by

$$\gamma_r = E\epsilon_1^{-r} = e^{-r\mu + \frac{r^2\sigma^2}{2}} \quad (r = 1, 2, \dots). \quad (19)$$

By Corollary 3.5 and (19), the moments β_r of Z may now be computed. One must require $r < 2\mu/\sigma^2$.

NOTE: If N is Normal with mean μ and variance σ^2 , then the r th moment of e^N is $E(e^N)^r = E(e^{rN}) = e^{r\mu + (1/2)r^2\sigma^2}$, from the well known formula for the moment generating function of the Normal distribution.

Example 3.2. (Pareto) Let $k > 0$, $\beta > 0$. Suppose $\epsilon_1 \sim \text{Pareto}(\beta, k)$ with density $f(x) = \beta k^\beta / x^{\beta+1} \mathbb{I}_{\{x \geq k\}}$. Let k and β be such that $E \log \epsilon_1 = \log k + \beta^{-1} > 0$. Then, one obtains $0 < \rho(x) < 1$ for every $x > c$ for $e^{-1/\beta} < k \leq 1$. Now the density function of $1/\epsilon_1$ is $f(y) = \beta k^\beta y^{\beta-1} \mathbb{I}_{\{0 < y \leq \frac{1}{k}\}}$, and its r th moment is provided by

$$E\epsilon_1^{-r} = \frac{\beta}{k^r(\beta + r)}. \quad (20)$$

Again, the moments of Z can be obtained in the same fashion as in Example 3.1.

Example 3.3. (Gamma) Assume $\epsilon_1 \sim \text{Gamma}(\alpha, \theta)$ with density

$$f(x) = \frac{\theta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\theta x} \mathbb{I}_{\{0 < x < \infty\}}, \quad \alpha, \theta > 0.$$

For $\alpha > \theta$, $E \log \epsilon_1 = \frac{\theta^\alpha}{\Gamma(\alpha)} \int_0^\infty (\log x) x^{\alpha-1} e^{-\theta x} dx > 0$. By Proposition 3.2-(b),(d), $d_1 = 0$, $d_2 = \infty$, and subsequently $0 < \rho(x) < 1$ for every $x > c$. The density function of $1/\epsilon_1$ is $f(y) = \frac{\theta^\alpha}{\Gamma(\alpha)} y^{-\alpha-1} e^{-\theta/y} \mathbb{I}_{\{0 < y < \infty\}}$, which is an Inverse-Gamma($\alpha, \frac{1}{\theta}$). Since the moment generating function of the Inverse-Gamma distribution does not exist, one can attain the finite moments by direct integration:

$$E\epsilon_1^{-r} = \frac{\theta^r \Gamma(\alpha - r)}{\Gamma(\alpha)}, \text{ or} \quad (21)$$

$$= \frac{\theta^r}{(\alpha - 1)(\alpha - 2) \cdots (\alpha - r)} \text{ for } \alpha \in \mathbb{Z}^+. \quad (22)$$

In the same manner, the moments of Z can be obtained.

3.3 Approximation to Survival Probability by Multiple Chebyshev Inequalities

With the moments of Z recursively obtained in Corollary 3.5, one obtains conservative estimates of ruin and survival probabilities using Chebyshev Inequality:

$$1 - \rho(x) = P(Z \geq x/c - 1) < \frac{\beta_r}{(x/c - 1)^r} \quad (r = 1, 2, \dots), \quad (x > c). \quad (23)$$

Notice that the smaller the upper estimate of ruin probability, the better the approximation of the true ruin probability is. Equivalently, the larger the lower estimate of survival probability, the better. Therefore, the estimate (23) with r over the one with $r + 1$ should be selected, iff

$$\frac{\beta_r}{(x/c - 1)^r} \leq \frac{\beta_{r+1}}{(x/c - 1)^{r+1}}, \quad \text{or } x \leq c(1 + \frac{\beta_{r+1}}{\beta_r}). \quad (24)$$

The upper estimate of ruin probability and the lower of survival probability, consequently, are obtained as follows:

$$1 - \rho(x) < \frac{\beta_r}{(x/c - 1)^r}, \quad \rho(x) > 1 - \frac{\beta_r}{(x/c - 1)^r}, \quad (x > c), \quad (25)$$

where r is chosen as follows:

$$\begin{cases} c(1 + \beta_r/\beta_{r-1}) < x \leq c(1 + \beta_{r+1}/\beta_r) & \text{if } r \geq 2, \\ c < x \leq c(1 + \beta_2/\beta_1) & \text{if } r = 1, \end{cases}$$

subject to the restriction $\gamma_r < 1$.

3.4 Numerical Examples

A conservative lower estimate of the survival probability using Chebyshev inequalities with different orders of moments of Z depending on x in (11) is numerically obtained in the following two examples. From these examples, one employs the empirical cumulative distribution function (ECDF), $\hat{F}(x)$, of $Z = \sum_{1 \leq n < \infty} (\epsilon_1 \epsilon_2 \cdots \epsilon_n)^{-1}$ to examine the performance of the estimate of survival probability using Chebyshev inequalities with different orders of

moments of Z depending on x . To obtain the empirical estimate of the distribution function of Z , one generates n random variables of ϵ_1^{-1} , where ϵ_1 is distributed by lognormal, or Pareto distribution in the examples. The sum of cumulative products of ϵ_1^{-1} 's yields a random variable, Z , whose distribution is obtained by replicating 3000 Z 's ($N = 3000$) in this section and the next. The empirical cumulative distribution (ECDF) of Z is then produced by $F_N(x) = \frac{1}{N} \sum_{j=1}^N \mathbb{I}\{Z_{n,j} = \sum_{i=1}^n \frac{1}{\epsilon_{1i}\epsilon_{2i}\dots\epsilon_{ni}} < x/c - 1\} = \frac{1}{N} \sum_{j=1}^N \mathbb{I}\{c(Z_{n,j} + 1) < x\}$ ($N = 3000$), i.e. the proportion of time that the agent survives until time n provided that the initial stock is $X_0 = x$. Since this empirical cumulative distribution will be close to the true probability ($\rho(x)$) for large enough simulations, we label $\hat{F}(x)$ as $\rho(x)$ from now on. Depending on the number of drawings of ϵ_1^{-1} , one can obtain either a finite-horizon survival probability ($\rho_n(x)$) or an infinite-horizon survival probability ($\rho(x)$).

Example 3.4. (*Lower estimates ($\underline{\rho(x)}$) of $\rho(x)$: Lognormal vs. Pareto.*)

Table 3¹ shows survival probabilities ($\rho(x)$) and the corresponding lower estimates ($\underline{\rho(x)}$) using multiple-Chebyshev inequalities with different orders of moments of Z depending on x , where ϵ_1 's are distributed by lognormal, Pareto, respectively. Since $\rho(x)$ of Pareto distribution reaches 1 immediately for $k > 1$ by Proposition 3.2, it is not of interest anymore. Instead, we only consider the case, where $e^{-1/\beta} < k \leq 1$ as described in Example 3.2, i.e. $0 < \rho(x) < 1$. Now, to compare the lower estimates of survival probabilities for the two distributions, the first and second moments of ϵ_1^{-1} are matched by having the parameters of Pareto distribution (β, k) free to select.

Table 1, Table 2 present some moments of ϵ_1^{-1} , and those of Z by recursive computation in Corollary 3.5, and the corresponding boundaries of x obtained by (25) for lognormal and Pareto distributions of ϵ_1 's, respectively. Both finite moments of Z , $\beta_r = EZ^r$, can be computed up to the point that $\gamma_r = E\epsilon_1^{-r}$ does not exceed 1. Table 1, Table 2 suggest that $\beta_r = EZ^r$ possibly decreases as far as $\gamma_r = E\epsilon_1^{-r} < 1$. However, β_{r+1}/β_r increases, so that the boundary value increases. For $e^{-1/\beta} < k < 1$, one can choose $k = 0.9$ and $\beta = 0.1$ such that $\epsilon_1 \sim \text{Pareto}(\beta = 0.1, k = 0.9)$. The parameters of the lognormal distribution are therefore taken as $\mu = 3.17$ and $\sigma^2 = 1.75$.

¹At least 100 of ϵ_1 's were drawn to have the series for Z converge. This is because the series for Z in the Pareto case converges more slowly than it does in the lognormal case.

The recursive computation to produce the moments for Z yields the result that for the lognormal case, $\gamma_4 = E\epsilon^{-4} > 1$ for the first time, which leads to $EZ^4 = \infty$ according to Corollary 3.5. Therefore, the lower estimate of survival probability using multiple-Chebyshev inequalities with different orders of moments of Z depending on x can be drawn only by the first and second moments of Z . Here, $\beta_3 = EZ^3$ is employed to get the boundary of x for $\rho(x) > 1 - \beta_2/(x/c - 1)^2$ in (25). For the Pareto case, $\gamma_{61} = E\epsilon^{-61} > 1$ for the first time, which results in $\beta_{61} = EZ^{61} = \infty$. Thus, the complete lower estimate of $\rho(x)$ by multiple-Chebyshev inequalities can be achieved by using 59 moments of Z . Similar to the lognormal case, $\beta_{60} = EZ^{60}$ is used to attain the boundary of x for $\rho(x) > 1 - \beta_{59}/(x/c - 1)^{59}$.

Table 3 exhibits the percentiles of survival probabilities and the corresponding lower estimates obtained by multiple-Chebyshev Inequalities with different orders of Z depending on x , where ϵ_1 's are respectively distributed by lognormal and Pareto distributions. Since the first and second moments (EZ , EZ^2) are matched for both distributions, the lower estimates obtained by those moments are basically the same as far as $x \leq 1.9481$, in Table 2. After that, the lower estimate of $\rho(x)$ for Pareto distribution becomes larger than that of $\rho(x)$ for lognormal distribution. That is because the remaining lower estimates of $\rho(x)$ for the Pareto case are obtained by Chebyshev inequalities with higher orders of moments of Z than the lower estimates of $\rho(x)$ for the lognormal case. Further, the survival probabilities for the Pareto case are lower than those for the lognormal case overall as x increases in Table 3. One can conclude that the lower estimate ($\underline{\rho(x)}$) by multiple-Chebyshev inequalities with different orders of moments of Z depending on x for the Pareto distribution case more closely approximates the corresponding survival probability ($\rho(x)$) than that of $\rho(x)$ for the lognormal distribution case does as x gets larger.

3.5 Finite-Horizon Survival Probability and Approximation by Multiple Chebyshev Inequalities

As in section 3.2 and 3.3, one can compute the moments of $Z_n := \sum_{1 \leq j \leq n} (\epsilon_1 \cdots \epsilon_j)^{-1}$ and achieve the conservative estimates of ruin and survival probabilities in finite time.

r	$E\epsilon_1^{-r}$	EZ^r	Boundaries
1	0.1010	0.1124	1.6808
2	0.0588	0.0765	6.0288
3	0.1971	0.3847	Inf

Table 1: $\ln \epsilon_1 \sim N(3.17, 1.75)$

Table 2: $\epsilon_1 \sim \text{Pareto}(\beta=0.1, k=0.9)$

x	1.1	1.2	1.4	1.6	1.8	2	2.2
lognormal	0.7193	0.8633	0.9513	0.9777	0.9863	0.9897	0.992
(lognormal)	0	0.4382	0.7191	0.8127	0.8805	0.9235	0.9469
Pareto	0.7723	0.8267	0.8913	0.9283	0.9553	0.9827	0.9963
(Pareto)	0	0.4382	0.7191	0.8127	0.8805	0.9275	0.9591

Table 3: Survival probabilities, $\rho(x)$, (lognormal, Pareto) vs lower estimates, $\underline{\rho}(x)$, by multiple-Chebyshev's inequalities with different moments of Z depending on x ((lognormal), (Pareto)) for $\ln \epsilon_1 \sim N(3.17, 1.75)$, $\epsilon_1 \sim \text{Pareto}(\beta=0.1, k=0.9)$.

Consider the derivation of $\rho(x)$ in (10)-(11). Then it is not hard to set up the probabilities of survival and ruin until time n . The survival probability of an economic agent up to the finite time n with an initial stock $x > c$ is

$$\begin{aligned}\rho_n(x) &:= P(X_n > c | X_0 = x) \\ &:= P(Z_n < x/c - 1).\end{aligned}\tag{26}$$

The finite-horizon ruin probability of an economic agent with an initial stock x is then $1 - \rho_n(x)$.

Now write $W_{n-1} := \sum_{2 \leq j \leq n} (\epsilon_1 \cdots \epsilon_j)^{-1}$. As the relation in Proposition 3.3, one can rewrite Z_n in terms of W_{n-1} and ϵ_1 :

$$Z_n \stackrel{\mathcal{L}}{=} \epsilon_1^{-1}(1 + W_{n-1}),\tag{27}$$

where W_{n-1} has the same distribution as Z_{n-1} , and W_{n-1} and ϵ_1 are independent. With this relation, for $E(\log \epsilon_1) > 0$, the moments of Z_n for all

$n = 1, 2, \dots$ are calculated recursively.

$$\beta_r^{(n)} = \gamma_r E(1 + W_{n-1})^r = \gamma_r \sum_{0 \leq j \leq r} \binom{r}{j} \beta_r^{(n-1)}, \quad (28)$$

where $\beta_r^{(n)} := EZ_n^r$, $\beta_0^{(n)} := 1$ for all $n \geq 1$, and $\beta_r^{(0)} := 0$ for all r . $\gamma_r := E(1/\epsilon_1)^r$ ($r = 1, 2, \dots$), and $\gamma_0 := 1$.

Example 3.5. According to the relation (28), one can obtain explicit forms of the moments of Z_n , $\beta_r^{(n)} = EZ_n^r$, for $n = 1, 2, \dots$:

$$\begin{aligned} \beta_r^{(1)} &= \gamma_r \quad (\forall r = 0, 1, 2, \dots); \\ \beta_1^{(2)} &= \gamma_1(1 + \beta_1^{(1)}) = \gamma_1(1 + \gamma_1), \\ \beta_2^{(2)} &= \gamma_2(1 + 2\beta_1^{(1)} + \beta_2^{(1)}) = \gamma_2(1 + 2\gamma_1 + \gamma_2), \\ \beta_3^{(2)} &= \gamma_3(1 + 3\beta_1^{(1)} + 3\beta_2^{(1)} + \beta_3^{(1)}) = \gamma_3(1 + 3\gamma_1 + 3\gamma_2 + \gamma_3), \dots; \\ \beta_1^{(3)} &= \gamma_1(1 + \beta_1^{(2)}) = \gamma_1(1 + \gamma_1(1 + \gamma_1)) = \gamma_1 + \gamma_1^2 + \gamma_1^3, \\ \beta_2^{(3)} &= \gamma_2(1 + 2\beta_1^{(2)} + \beta_2^{(2)}) = \gamma_2(1 + 2\gamma_1 + 2\gamma_1^2 + \gamma_2 + 2\gamma_1\gamma_2 + \gamma_2^2), \dots; \\ &\vdots \end{aligned}$$

Finally the upper estimate of ruin probability and the lower of survival probability until time n are obtained as follows:

$$1 - \rho_n(x) < \frac{\beta_r^{(n)}}{(x/c - 1)^r}, \quad \rho_n(x) > 1 - \frac{\beta_r^{(n)}}{(x/c - 1)^r}, \quad (x > c), \quad (29)$$

where r is chosen as follows:

$$\begin{cases} c(1 + \beta_r^{(n)}/\beta_{r-1}^{(n)}) < x \leq c(1 + \beta_{r+1}^{(n)}/\beta_r^{(n)}) & \text{if } r \geq 2, \\ c < x \leq c(1 + \beta_2^{(n)}/\beta_1^{(n)}) & \text{if } r = 1, \end{cases}$$

Example 3.6. (*Lower estimates ($\rho_n(x)$) of finite-horizon survival probabilities ($\rho_n(x)$): Lognormal, Pareto, and Gamma.*) Table 4-Table 6 demonstrate boundary values of x based on (29), where ϵ_1 's are distributed by

lognormal, Pareto, and gamma distributions, respectively. In the same fashion as in Example 3.4, the first and second moments of ϵ_1^{-1} 's are matched for all the three distributions, so that one can compare the survival probabilities ($\rho_n(x)$), with the corresponding lower bounds ($\underline{\rho}_n(x)$) by multiple-Chebyshev inequalities with different moments of Z_n depending on x . In such a way, one gets $\ln \epsilon_1 \sim N(0.2146, 0.0645)$, $\epsilon_1 \sim \text{Pareto}(\beta=3, k=0.9)$, and $\epsilon_1 \sim \text{Gamma}(\alpha=17, \theta=13.3333)$. The finite moments of Z_n are calculated recursively from these distributions. These tables show that the boundaries of x increase for $\forall r \geq 1$ as n increases, i.e., $Z_n \rightarrow Z$.

r	Z_3	Z_5	Z_{10}	Z_{20}	Z
c	1	1	1	1	1
1	3.3137	4.3807	5.9908	7.0502	7.2857
2	3.546	4.8419	7.0433	8.8004	9.2795
3	3.8072	5.3915	8.4725	11.6162	12.7826
4	4.1018	6.0525	10.4814	16.7021	20.5384
5	4.4353	6.8551	13.4176	27.5237	52.1729

Table 4: Boundaries of x for the lower estimates of $\rho_n(x)$ by multiple-Chebyshev inequalities with r^{th} moments of Z_n ($n = 3, 5, 10, 20$) and Z , where $\ln \epsilon_1 \sim N(0.2146, 0.0645)$, and $c=1$ (a fixed amount of consumption).

r	Z_3	Z_5	Z_{10}	Z_{20}	Z
c	1	1	1	1	1
1	3.3137	4.3807	5.9908	7.0502	7.2857
2	3.4698	4.6962	6.7183	8.2603	8.6618
3	3.5932	4.9592	7.3887	9.5165	10.1737
4	3.6938	5.1826	8.0083	10.8238	11.8661
5	3.7777	5.3752	8.5816	12.1805	13.791

Table 5: Boundaries of x for the lower estimates of $\rho_n(x)$ by multiple-Chebyshev's inequalities with r^{th} moments of Z_n ($n = 3, 5, 10, 20$) and Z , where $\epsilon_1 \sim \text{Pareto}(\beta=3, k=0.9)$, and $c=1$ (a fixed amount of consumption).

Table 7-Table 9 demonstrate survival probabilities ($\rho_n(x)$) in finite time and the corresponding lower estimates ($\underline{\rho}_n(x)$) using multiple-Chebyshev inequalities with different moments of Z_n depending on x . These Tables demonstrate

r	Z_3	Z_5	Z_{10}	Z_{20}	Z
c	1	1	1	1	1
1	3.3137	4.3807	5.9908	7.0502	7.2857
2	3.5606	4.87	7.1073	8.9091	9.405
3	3.8589	5.4978	8.7526	12.201	13.546
4	4.2255	6.3255	11.3481	19.2233	25.1935
5	4.6846	7.4534	15.8266	39.6022	256.6073

Table 6: Boundaries of x for the lower estimates of $\rho_n(x)$ by multiple-Chebyshev inequalities with r^{th} moments of Z_n ($n = 3, 5, 10, 20$) and Z , where $\epsilon_1 \sim \Gamma(\alpha=17, \theta=13.3333)$, and $c=1$ (a fixed amount of consumption) (Computations in Tables 4-6 corresponding to any $c \neq 0$ can be handled by interpreting x as x/c).

the way the survival probabilities get lower as time n increases when x fixed. They also display how much higher the survival probabilities get for each time n as an initial stock x increases. In addition to these observations, the estimates, $\rho_n(x)$, by multiple-Chebyshev inequalities with different orders of moments of Z_n depending on x become closer to $\rho_n(x)$ as x becomes larger for each n . In particular, one can observe that the survival probabilities $\rho_n(x)$ for Pareto distribution are more closely approximated by the estimates $\rho_n(x)$ than those for the other two distributions as x gets larger for each n . Moreover, the lower estimates $\rho_n(x)$ for lognormal distribution approximate the survival probabilities $\rho_n(x)$ slightly better than those for gamma distribution do. It turns out that the lower estimates of $\rho_n(x)$ by multiple-Chebyshev inequalities with different orders of moments of Z_n depending on x for Pareto distribution, lognormal, and gamma in this order show good performance of approximating the true probabilities.

x	$\rho_{l,3}(x)$	$\rho_3(x)$	$\rho_{l,5}(x)$	$\rho_5(x)$	$\rho_{l,10}(x)$	$\rho_{10}(x)$	$\rho_{l,20}(x)$	$\rho_{20}(x)$
3.5	0.2202	0.7530	0.0000	0.3633	0.0000	0.1290	0.0000	0.0907
7.5	0.9907	0.9997	0.9257	0.9927	0.5396	0.8890	0.3027	0.8193
9.5			0.9806	0.9997	0.8190	0.9700	0.6258	0.9280
12.5			0.9957	1.0000	0.9555	0.9963	0.8605	0.9807

Table 7: The lower estimates of $\rho_n(x)$ using multiple-Chebyshev inequalities ($\rho_{l,n}(x)$) vs the survival probabilities ($\rho_n(x)$) for $\ln \epsilon_1 \sim N(0.2146, 0.0645)$

x	$\rho_{l,3}(x)$	$\rho_3(x)$	$\rho_{l,5}(x)$	$\rho_5(x)$	$\rho_{l,10}(x)$	$\rho_{10}(x)$	$\rho_{l,20}(x)$	$\rho_{20}(x)$
3.5	0.2296	0.7070	0.0000	0.3557	0.0000	0.1713	0.0000	0.1393
7.5	1.0000	1.0000	0.9976	1.0000	0.5718	0.8783	0.3027	0.7930
9.5					0.8972	0.9770	0.6517	0.9323
12.5					0.9961	0.9990	0.9135	0.9853

Table 8: The lower estimates of $\rho_n(x)$ using multiple-Chebyshev inequalities ($\rho_{l,n}(x)$) vs the survival probabilities ($\rho_n(x)$) for $\epsilon_1 \sim \text{Pareto}(\beta=3, k=0.9)$

x	$\rho_{l,3}(x)$	$\rho_3(x)$	$\rho_{l,5}(x)$	$\rho_5(x)$	$\rho_{l,10}(x)$	$\rho_{10}(x)$	$\rho_{l,20}(x)$	$\rho_{20}(x)$
3.5	0.2202	0.7860	0.0000	0.3653	0.0000	0.1293	0.0000	0.0780
7.5	0.9901	0.9997	0.9192	0.9887	0.5347	0.9133	0.3027	0.8267
9.5			0.9848	0.9983	0.8102	0.9767	0.6206	0.9293
12.5			0.9983	1.0000	0.9490	0.9957	0.8508	0.9777

Table 9: The lower estimates of $\rho_n(x)$ using multiple-Chebyshev inequalities ($\rho_{l,n}(x)$) vs the survival probabilities ($\rho_n(x)$) for $\epsilon_1 \sim \Gamma(\alpha=17, \theta=13.3333)$

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